

MTH250 - CALCULUS

Lecture No. 31

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by H. Anton, I Bivens and S. Davis, John Wiley & Sons, 10th edition, 2012.

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LECTURE 31

Green's and Stokes' theorems

31.1 Orientation of non-parametric surfaces

Remark that each of the equations

$$z - g(x, y) = 0, \quad y - g(x, z) = 0, \quad x - g(y, z) = 0$$

has the form $G(x, y, z) = 0$ and hence can be viewed as a level surface of the function $G(x, y, z)$.

Since gradient of G is normal to the level surface, it follows that the unit normal $\mathbf{n}$ is either

$$\frac{\nabla G}{|\nabla G|} \quad \text{or} \quad -\frac{\nabla G}{|\nabla G|}.$$

Thus in all three cases $\mathbf{n} = \frac{\nabla G}{|\nabla G|}$.

Theorem 31.1. *Let σ be a smooth surface of the form $z = g(x, y)$, $y = g(x, z)$, or $x = g(y, z)$, and suppose that the component functions of the vector field $\mathbf{F}$ are continuous on σ . Suppose also that the equation for σ is rewritten as $G(x, y, z) = 0$ by taking g to the left side of the equation, and let R be the projection of σ on the coordinate plane determined by the independent variables of g . If σ has positive orientation, then*

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \nabla G dA.$$

$z = g(x, y)$	$y = g(z, x)$	$x = g(y, z)$
 $\mathbf{n} = \frac{-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$ Positive k-component Positive orientation	 $\mathbf{n} = \frac{-\frac{\partial y}{\partial x}\mathbf{i} + \mathbf{j} - \frac{\partial y}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}}$ Positive j-component Positive orientation	 $\mathbf{n} = \frac{\mathbf{i} - \frac{\partial x}{\partial y}\mathbf{j} - \frac{\partial x}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}}$ Positive i-component Positive orientation
 $-\mathbf{n} = \frac{\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$ Negative k-component Negative orientation	 $-\mathbf{n} = \frac{\frac{\partial y}{\partial x}\mathbf{i} - \mathbf{j} + \frac{\partial y}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}}$ Negative j-component Negative orientation	 $-\mathbf{n} = \frac{-\mathbf{i} + \frac{\partial x}{\partial y}\mathbf{j} + \frac{\partial x}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}}$ Negative i-component Negative orientation

Figure 31.1: Surface area.

If $z = g(x, y)$, then we have $G(x, y, z) = z - g(x, y)$, so

$$\nabla G = -\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k}$$

and taking R to be the projection of the surface $z = g(x, y)$ on the xy -plane yields

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \left(-\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k} \right) dA$$

if σ is oriented up and

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} - \hat{k} \right) dA$$

if σ is oriented down.

Example 31.1. Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy -plane and suppose that σ is oriented up. Find the flux of the vector field $\mathbf{F}(x, y) = x\hat{i} + y\hat{j} + z\hat{k}$ across σ .

Solution.

$$\begin{aligned}
 \Phi &= \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \left(-\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k} \right) dA \\
 &= \iint_R (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2x\hat{i} + 2y\hat{j} + \hat{k}) dA \\
 &= \iint_R (x^2 + y^2 + 1) dA = \int_0^{2\pi} \int_0^1 (r^2 + 1) r dr d\theta \\
 &= \int_0^{2\pi} \left(\frac{3}{4} \right) d\theta = \frac{3\pi}{2}.
 \end{aligned}$$

□

31.2 Green's theorem for line integrals

Theorem 31.2 (Green's).

Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing R , then

$$\int_C f(x, y)dx + g(x, y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Thus, we have the following three formulae that express the area A of a region R in terms of line integrals around the boundary:

$$A = \oint_C x dy = - \oint_C y dx = \frac{1}{2} \oint_C -y dx + x dy$$

where by an integral sign with a superimposed circle we denote a line integral around a simple closed curve.

Example 31.2. Using Green's theorem to evaluate $\int_C x^2 y dx + x dy$ along the triangular path shown in Figure 31.2

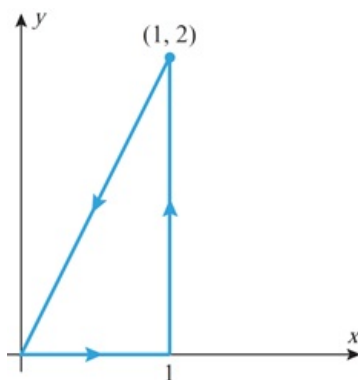


Figure 31.2: Triangular path.

Solution. Since $f(x, y) = x^2 y$ and $g(x, y) = x$, it follows from Green's theorem that

$$\begin{aligned} \int_C x^2 y dx + x dy &= \iint_R \left[\frac{\partial x}{\partial x}(x) - \frac{\partial}{\partial y}(x^2 y) \right] dA = \int_0^1 \int_0^{2x} (1 - x^2) dy dx \\ &= \int_0^1 (2x - 2x^2) dx = \left[x^2 - \frac{x^3}{3} \right]_0^1 = \frac{1}{3}. \end{aligned}$$

□

Example 31.3. Use a line integral to find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. The ellipse, with counterclockwise orientation, can be represented parametrically by

$$x = a \cos t, \quad y = b \sin t \quad (0 \leq t \leq 2\pi).$$

If we denote this curve by C , then the area A enclosed by the ellipse is

$$\begin{aligned} A &= \frac{1}{2} \oint_C -y dx + x dy \\ &= \frac{1}{2} \int_0^{2\pi} [(-b \sin t)(-a \sin t) + (a \cos t)(b \cos t)] dt \\ &= \frac{1}{2} ab \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \frac{1}{2} ab \int_0^{2\pi} dt = \pi ab. \end{aligned}$$

□

31.2.1 Green's function for piecewise smooth regions

We consider a multiply connected region R and divide it into two regions R' and R'' . Suppose $f(x, y)$ and $g(x, y)$ have continuous first order partial derivatives on some region containing R and satisfy the hypothesis of Green's theorem. Then,

$$\begin{aligned} \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA &= \iint_{R'} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA + \iint_{R''} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA. \\ &= \oint_{\partial R'} f(x, y) dx + g(x, y) dy + \oint_{\partial R''} f(x, y) dx + g(x, y) dy. \end{aligned}$$

where $\partial R'$ and $\partial R''$ denote the boundaries of the regions R' and R'' respectively.

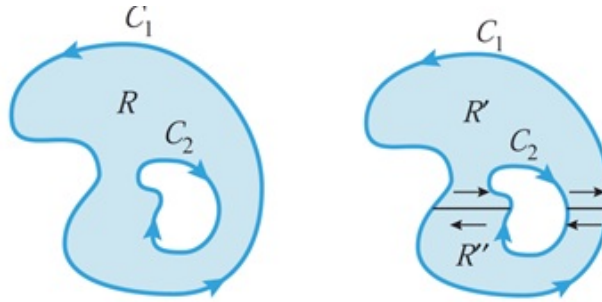


Figure 31.3: Piecewise smooth regions.

31.3 Stokes' theorem

Definition 31.3 (Divergence and Curl).

If $\mathbf{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k}$, then we define the divergence of $\mathbf{F}$, written $\text{div}\mathbf{F}$, to be the function given by

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

We define the curl of $\mathbf{F}$, written $\text{curl}\mathbf{F}$, to be the vector field given by

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{pmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}.$$

Theorem 31.4 (Stokes'). Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the components of the vector field $\mathbf{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k}$, are continuous and have continuous first partial derivatives on some open set containing σ , and if $\mathbf{T}$ is the unit tangent vector to C , then

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_{\sigma} (\text{curl}\mathbf{F}) \cdot \mathbf{n} dS.$$

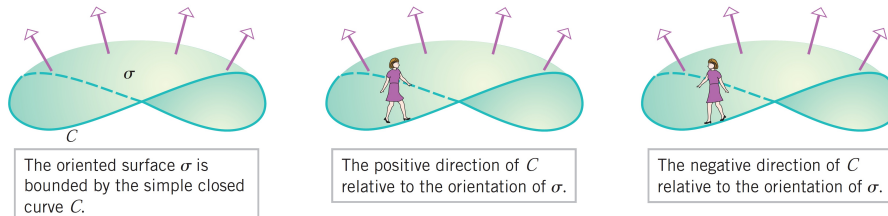


Figure 31.4: Orientations of the surfaces.

Remark 31.5. Stokes' theorem states that: The line integral around the boundary curve of σ of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Example 31.4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orientation of C to be counterclockwise when viewed from above.)

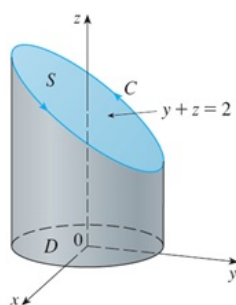


Figure 31.5: Curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution. The integral can be evaluated directly. However, it's easier to use Stokes' Theorem. First we compute curl of $\mathbf{F}$.

$$\text{curl}\mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\hat{k}.$$

If we orient S upward, C has the induced positive orientation. The projection D of S on the xy -plane is the disk $x^2 + y^2 \leq 1$.

Let $z = g(x, y) = 2 - y$, then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D (1 - 2y) dA \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r dr d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin \theta \right] d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta = \frac{1}{2}(2\pi) + 0 = \pi. \end{aligned}$$

□

In general, if σ_1 and σ_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' theorem, then

$$\iint_{\sigma_1} \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\sigma_2} \text{curl}\mathbf{F} \cdot d\mathbf{S}.$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

31.3.1 Relationship between Green's and Stokes' theorems

We can express a vector field

$$\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$$

in 2D as a vector field in 3D by

$$\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j} + 0\hat{k}$$

If R is a region in the xy -plane enclosed by a simple, closed, piecewise smooth curve C , then we can treat R as a flat surface, and we treat a surface integral over R as an ordinary double integral over R . Thus, if we orient C up and C counterclockwise looking down the positive z -axis, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma \text{curl} \mathbf{F} \cdot \hat{k} dA.$$

Since $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} = 0$. Therefore,

$$\text{curl} \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = -\frac{\partial g}{\partial z}\hat{i} + \frac{\partial f}{\partial z}\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

Substituting this expression in previous equation and expressing the integrals in terms of components yields

$$\oint_C f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \hat{k} dA$$

Hence, we conclude that Green's theorem is a special case of Stokes' theorem and can be regarded as a higher-dimensional version of Green's Theorem.

Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve while Stokes' theorem relates a surface integral over a surface σ to a line integral around the boundary curve of S (a space curve).